THE SUPPORT OF THE LOGARITHMIC EQUILIBRIUM MEASURE ON SETS OF REVOLUTION IN $\mathbb{R}^3$

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Abstract. For surfaces of revolution B in $\mathbb{R}^3$, we investigate the limit distribution of minimum energy point masses on B that interact according to the logarithmic potential $\log(1/r)$, where r is the Euclidean distance between points. We show that such limit distributions are supported only on the "out-most" portion of the surface (e.g., for a torus, only on that portion of the surface with positive curvature). Our analysis proceeds by reducing the problem to the complex plane where a non-singular potential kernel arises whose level lines are ellipses.

1. Introduction

For a collection of $N(\geq 2)$ distinct points $\omega_N := \{x_1, \ldots, x_N\} \subset \mathbb{R}^3$ and $s > 0$, the *Riesz s-energy of* ω_N is defined by

$$E_s(\omega_N) := \sum_{1\leq i\neq j\leq N} k_s(x_i, x_j) = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} k_s(x_i, x_j),$$

where, for $x, y \in \mathbb{R}^3$, $k_s(x, y) := 1/|x-y|^s$. As $s \to 0$, it is easily verified that

$$(k_s(x, y) - 1)/s \to \log\left(1/|x-y|\right)$$

and so it is natural to define $k_0(x, y) := \log\left(1/|x-y|\right)$. For a compact set $B \subset \mathbb{R}^3$ and $s \geq 0$, the *N-point s-energy of B* is defined by

$$\mathcal{E}_s(B, N) := \inf\{E_s(\omega_N) \mid \omega_N \subset B, |\omega_N| = N\}, \tag{1}$$

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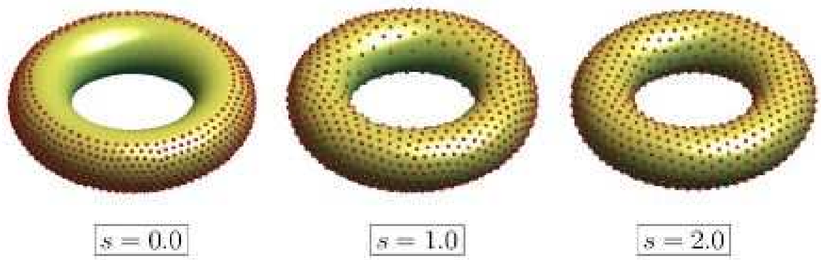

FIGURE 1. Near optimal Riesz s-energy configurations ($N = 1000$ points) on a torus in $\mathbb{R}^3$ for $s = 0, 1$, and 2.

where $|X|$ denotes the cardinality of a set X. Note that the logarithmic ($s = 0$) minimum energy problem is equivalent to the maximization of the product

$$\prod_{1 \le i \neq j \le N} |x_i - x_j|,$$

and that for planar sets, such optimal points are known as *Fekete points*. (The fast generation of near optimal logarithmic energy points for the sphere S^2 is the focus of one of S. Smale's "mathematical problems for the next century"; see [14].)

If $0 \le s < \dim B$ (the Hausdorff dimension of B), the limit distribution (as $N \to \infty$) of optimal N-point configurations is given by the *equilibrium measure* $\lambda_{s,B}$ that minimizes the continuous energy integral

$$I_s(\mu) := \iint_{B \times B} k_s(x, y)\, d\mu(x)\, d\mu(y)$$

over the class $\mathcal{M}(B)$ of (Radon) probability measures μ supported on B. In addition, the asymptotic order of the Riesz s-energy is N^2; more precisely we have $\mathcal{E}_s(B, N)/N^2 \to I_s(\lambda_{s,B})$ as $N \to \infty$ (cf. [11, Section II.3.12]). In the case when $B = S^2$, the unit sphere in $\mathbb{R}^3$, the equilibrium measure is simply the normalized surface area measure. If $s \ge \dim B$, then $I_s(\mu) = \infty$ for every $\mu \in \mathcal{M}(B)$ and potential theoretic methods cannot be used. However, it was recently shown in [7] that when B is a d-rectifiable manifold of positive d-dimensional Hausdorff measure and $s \ge d$, optimal N-point configurations are uniformly distributed (as $N \to \infty$) on B with respect to d-dimensional Hausdorff measure restricted to B. (The assertion for the case $s = d$

FIGURE 2. Minimum logarithmic energy points on various toroidal surfaces.

further requires that B be a subset of a C^1 manifold.) For further extensions of these results, see [3]. Related results and applications appear in [5] (coding theory), [13] (cubature on the sphere), and [1] (finite normalized tight frames).

In Figure 1, we show near optimal Riesz s-energy configurations for the values of $s = 0, 1,$ and 2 for $N = 1000$ points restricted to live on the torus B obtained by revolving the circle of radius 1 and center $(3, 0)$ about the y-axis. (For recent results on the disclinations of minimal energy points on toroidal surfaces, see [4].) The somewhat surprising observation that there are no points on the "inner" part of the torus in the case $s = 0$ (and, in fact, as well for s near 0) is what motivated us to investigate the support of the logarithmic equilibrium measure $\lambda_{0,B}$. In this paper we show that, in fact, this is a general phenomenon for optimal logarithmic energy configurations of points restricted to sets of revolution in $\mathbb{R}^3$ (see Figure 2).

2. Preliminaries

In this paper we focus on the logarithmic kernel k_0. Let $B \subset \mathbb{R}^3$ be compact. As in the previous section, the *logarithmic energy* of a measure $\mu \in \mathcal{M}(B)$ is given by

$$I_0(\mu) = \iint_{B\times B} \log \frac{1}{|p-q|}\, d\mu(p) d\mu(q) \tag{2}$$

and the corresponding *potential* U^μ is defined by

$$U^\mu(p) := \int_B \log \frac{1}{|p-q|}\, d\mu(q) \qquad (p \in \mathbb{R}^3). \tag{3}$$

Let $V_B := \inf_{\mu\in\mathcal{M}(B)} I_0(\mu)$. The *logarithmic capacity of* B, denoted by $\mathrm{cap}(B)$, is $\exp(-V_B)$. A condition $C(p)$ is said to hold *quasi-everywhere* on B if it holds for all $p \in B$ except for a subset of logarithmic capacity zero.[1] If $\mathrm{cap}(B) > 0$, then there is a unique probability measure $\mu_B \in \mathcal{M}(B)$ (called the *equilibrium measure on* B) such that $I(\mu_B) = V_B$ (this is implicit in the references [11, 12]). Furthermore, the equality $U^{\mu_B}(p) = V_B$ holds quasi-everywhere on the support of μ_B and $U^{\mu_B}(p) \geq V_B$ quasi-everywhere on B.

We now turn our attention to sets of revolution in $\mathbb{R}^3$. Let $\mathbb{R}_+ := [0,\infty)$ and, for $t \in [0, 2\pi)$, let $\sigma_t : \mathbb{R}^3 \to \mathbb{R}^3$ denote the rotation about the y-axis through an angle t:

$$\sigma_t(x, y, \zeta) = (x \cos t - \zeta \sin t, y, x \sin t + \zeta \cos t).$$

For a compact set A contained in the right half-plane $H^+ := \mathbb{R}_+ \times \mathbb{R}$, let $\Gamma(A) \subset \mathbb{R}^3$ be the set obtained by revolving A around the y-axis, that is,

$$\Gamma(A) := \{\sigma_t(x, y, 0) \mid (x, y) \in A,\ 0 \leq t < 2\pi\}.$$

We say that $A \subset H^+$ is *non-degenerate* if $\mathrm{cap}\,(\Gamma(A))$ is positive. For example, if A contains at least one point not on the y-axis, then A is non-degenerate.

3. Reduction to the xy-plane

A Borel measure $\tilde\nu \in \mathcal{M}(\mathbb{R}^3)$ is *rotationally symmetric about the* y*-axis* if $\tilde\nu = \tilde\nu \circ \sigma_t$ for all $t \in [0, 2\pi)$. If $\tilde\nu$ is rotationally symmetric about the y-axis, then $d\tilde\nu = \frac{1}{2\pi} dt d\nu$, where $\nu := \tilde\nu \circ \Gamma \in \mathcal{M}(H^+)$ and dt denotes Lebesgue measure on $[0, 2\pi)$. Identifying points $z, w \in H^+$ as complex numbers $z = x + iy = (x, y, 0)$ and $w = u + iv = (u, v, 0)$ we have

$$\begin{aligned} I_0(\tilde\nu) &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \log\frac{1}{|p-q|}\, d\tilde\nu(p) d\tilde\nu(q) \\ &= \iint_{H^+\times H^+} K(z, w)\, d\nu(z) d\nu(w) \qquad (4) \\ &=: J(\nu), \end{aligned}$$

where

$$K(z, w) := \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|\sigma_t(z) - w|}\, dt. \qquad (5)$$

[1]The logarithmic capacity of a Borel set E is the sup of the capacities of its compact subsets. Any set that is contained in a Borel set of capacity zero is said to have capacity zero.

Notice that

$$
\begin{aligned}
(6) \qquad |\sigma_t(z) - w|^2 &= (x\cos t - u)^2 + (y-v)^2 + x^2\sin^2 t \\
&= x^2 + u^2 + (y-v)^2 - 2xu\cos t.
\end{aligned}
$$

Let $w_* := -u + iv = -\overline{w}$ denote the reflection of w in the y-axis. Then, using (6) and the formula

$$
\frac{1}{2\pi}\int_0^{2\pi} \log(a + b\cos t)\, dt = \log\frac{a + \sqrt{a^2 - b^2}}{2}
$$

with $a = (y-v)^2 + x^2 + u^2$ and $b = -2xu$, we obtain

$$
(7) \qquad K(z,w) = -\frac{1}{2}\log\frac{a+\sqrt{a^2-b^2}}{2} = \log\frac{2}{|z-w| + |z-w_*|},
$$

where we have used

$$
2\left(a + \sqrt{a^2-b^2}\right) = \left(\sqrt{a+b} + \sqrt{a-b}\right)^2 = \left(|z-w| + |z-w_*|\right)^2.
$$

3.1. **Equilibrium measure $\lambda_A \in \mathcal{M}(A)$.** For a non-degenerate compact set $A \subset H^+$, the uniqueness of the equilibrium measure $\mu_{\Gamma(A)}$ and the symmetry of the revolved set $\Gamma(A)$ imply that $\mu_{\Gamma(A)}$ is rotationally symmetric about the y-axis and so $d\mu_{\Gamma(A)} = \frac{1}{2\pi} dt d\lambda_A$, where for any Borel set $B \subset H^+$

$$
(8) \qquad \lambda_A(B) := \mu_{\Gamma(A)}(\Gamma(B)).
$$

Furthermore, if $\nu \in \mathcal{M}(A)$, then $d\tilde{\nu} := \frac{1}{2\pi} dt d\nu$ is rotationally symmetric about the y-axis and so we have

$$
J(\lambda_A) \geq \inf_{\nu\in\mathcal{M}(A)} J(\nu) = \inf_{\nu\in\mathcal{M}(A)} I_0(\tilde{\nu}) \geq I_0(\mu_{\Gamma(A)}) = J(\lambda_A),
$$

which leads to the following proposition.

Proposition 1. *Suppose A is a non-degenerate compact set in H^+ and let $\lambda_A \in \mathcal{M}(A)$ be defined by (8). Then λ_A is the unique measure in $\mathcal{M}(A)$ that minimizes $J(\nu)$ over all measures $\nu \in \mathcal{M}(A)$. That is, λ_A is the* equilibrium measure *for the kernel K and set A.*

For $\nu \in \mathcal{M}(A)$, we define the *(K-)potential* W^ν by

$$
\begin{aligned}
(9) \qquad W^\nu(z) &:= \int_A K(z,w)\, d\nu(w) \\
&= \int_A \log\frac{2}{|z-w| + |z-w_*|} d\nu(w) \qquad (z \in H^+).
\end{aligned}
$$

Then, for $z = (x, y, 0) \in H^+$, we have

$$\begin{aligned} U^{\mu_{\Gamma(A)}}(z) &= \int_{\Gamma(A)} \log \frac{1}{|z-q|}\, d\mu_{\Gamma(A)}(q) \\ &= \frac{1}{2\pi} \int_A \int_0^{2\pi} \log \frac{1}{|z-\sigma_t(w)|}\, dt\, d\lambda_A(w) \\ &= \int_A K(z,w)\, d\lambda_A(w) = W^{\lambda_A}(z). \end{aligned}$$

From the properties of $U^{\mu_{\Gamma(A)}}$, we then infer the following lemma.

Lemma 2. *Suppose A is a non-empty compact set in the interior of H^+. Let λ_A be the equilibrium measure for A with respect to the kernel K. Then the potential W^{λ_A} satisfies $W^{\lambda_A}(z) = J(\lambda_A)$ for z in the support of λ_A and $W^{\lambda_A}(z) \geq J(\lambda_A)$ for $z \in A$.*

Remark: In Lemma 2 we no longer need a quasi-everywhere exceptional set, since each point of A generates a circle in $\mathbb{R}^3$ with positive logarithmic capacity.

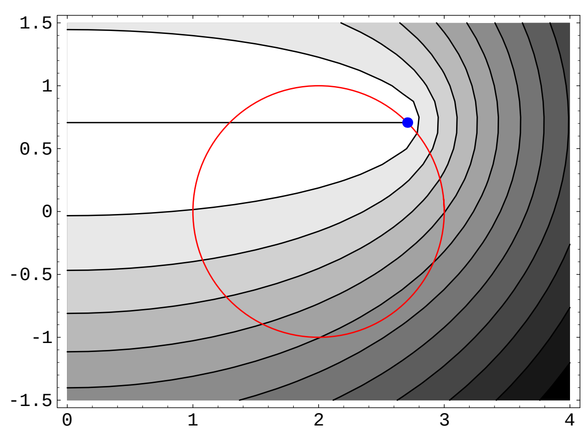

FIGURE 3. Level curves for $K(z, w)$ for w a fixed point on the unit circle centered at (2,0).

3.2. **Properties of K.** Let $s(z, w) := |z - w| + |z - w_*|$. Then $K(z, w) = -\log(s(z, w)/2)$ and so, for fixed $w \in H^+$, the level sets of $K(\cdot, w)$ are ellipses with foci w and w_* as shown in Figure 3. Since the foci have the same imaginary part $v = \mathrm{Im}[w] = \mathrm{Im}[w_*]$, it follows from geometrical considerations that $K(\cdot, w)$ is strictly decreasing along horizontal rays $[iy, \infty + iy)$ for $y \neq v$. Along the horizontal ray

$[iv, \infty + iv)$, we have that $K(\cdot, w)$ is constant on the line segment $[iv, w]$ and strictly decreasing on the ray $[w, \infty + iv)$.

Furthermore, K is clearly continuous at any $(z, w) \in H^+ \times H^+$ unless $z = w = iy$ for some $y \in \mathbb{R}$. Since $|z - w_*| = |(z - w_*)_*| = |w - z_*|$, it follows that K is symmetric, that is, $K(z, w) = K(w, z)$ for $z, w \in H^+$. We summarize these properties of K in the following lemma.

Lemma 3. *The kernel $K : H^+ \times H^+ \to \mathbb{R}$ in (7) has the following properties:*

(a) *K is symmetric: $K(z, w) = K(w, z)$ for $w, z \in H^+$.*
(b) *K is continuous at all points $(z, w) \in H^+ \times H^+$ except points (z, z) such that $\mathrm{Re}(z) = 0$.*
(c) *Let $u \geq 0$ and $y \neq v \in \mathbb{R}$ be fixed. Then $K(x + iy, u + iv)$ is a strictly decreasing function of x for $x \in [0, \infty)$. Furthermore, $K(x + iy, u + iy)$ is constant for $x \in [0, u]$ and is strictly decreasing for $x \in [u, \infty)$.*

The following lemma is then a consequence of Lemma 3.

Lemma 4. *Suppose $\nu \in \mathcal{M}(A)$ is not a point mass (that is, the support of ν contains at least two points). Then the potential $W^\nu(z)$ is strictly decreasing along the horizontal rays $[iy, \infty + iy)$ for all $y \in \mathbb{R}$.*

If A is a non-degenerate compact set in H^+, let $P(A)$ denote the projection of the set A onto the y-axis and for $y \in P(A)$, define $x_A(y) = \max\{x \mid (x, y) \in A\}$. We then let A_+ denote the "right-most" portion of A, that is,

$$A_+ := \{(x_A(y), y) \mid y \in P(A)\}.$$

Using Lemmas 2 and 4 we then obtain the following result.

Theorem 5. *Suppose A is a compact set in H^+ such that A_+ is contained in the interior of H^+. Then the support of the equilibrium measure $\lambda_A \in \mathcal{M}(A)$ is contained in A_+.*

4. Convexity

Recall that a function $f : [a, b] \to \mathbb{R}$ is *strictly convex on* $[a, b]$ if $f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)$ for all $a \leq x < y \leq b$ and $0 < \theta < 1$.

Theorem 6. *Suppose A is a compact set in H^+ such that A_+ is contained in the interior of H^+ and $\gamma : [a, b] \to H^+$ is continuous. Further suppose that*

(a) *$A_+ \subset \gamma^* := \{\gamma(s) \mid a \leq s \leq b\}$ and*

(b) $K(\gamma(\cdot),\gamma(s))$ *is a strictly convex function on the intervals* $[a,s]$ *and* $[s,b]$ *for each fixed* $s\in[a,b]$.

Then there is some closed interval $I\subset[a,b]$ *such that* $\operatorname{supp}\lambda_A=\gamma(I)\cap A_+$.

Proof. Suppose A and γ satisfy (a) and (b). From Theorem 5 we have $\operatorname{supp}\lambda_A\subset\gamma^*$. Let $t_1:=\min_{a\le t\le b}\{t\mid\gamma(t)\in\operatorname{supp}\lambda_A\}$ and $t_2:=\max_{a\le t\le b}\{t\mid\gamma(t)\in\operatorname{supp}\lambda_A\}$. Suppose that G is an open interval in $I:=[t_1,t_2]$ such that $\gamma(G)\cap\operatorname{supp}\lambda_A=\emptyset$. Then $W^{\lambda_A}\circ\gamma$ is strictly convex on G and $W^{\lambda_A}(z)=J(\lambda_A)$ for $z\in\operatorname{supp}\lambda_A$ and so we have $W^{\lambda_A}(\gamma(t))<J(\lambda_A)$ for $t\in G$. Hence, Lemma 2 implies that $\gamma(G)\cap A=\emptyset$ which then implies $\operatorname{supp}\lambda_A=\gamma(I)\cap A_+$. □

We next consider several examples where we can verify that the hypotheses of Theorem 6 hold. In these examples, γ is a smooth curve, but note that A_+ is only required to be a compact subset of γ^*. For example, A_+ may be a Cantor subset of γ^*.

We first consider a case where we can completely specify the support of λ_A.

Corollary 7. *Suppose A is a non-degenerate compact subset in H^+ such that A_+ is contained in a vertical line segment $[R+ci,R+di]$ for some $R>0$. Then* $\operatorname{supp}\lambda_A=A_+$.

Proof. Consider the parametrization $\gamma(t)=R+it$, $c\le t\le d$, of the line segment $[R+ci,R+di]$. For $s,t\in[c,d]$, $s\ne t$, direct calculation shows $K(\gamma(t),\gamma(s))=-\log(|s-t|+\sqrt{4R^2+(s-t)^2})+\log 2$ and

$$\frac{d}{dt}K(\gamma(t),\gamma(s)) = \frac{\operatorname{sgn}(s-t)}{\sqrt{4R^2+(s-t)^2}}, \tag{10}$$

$$\frac{d^2}{dt^2}K(\gamma(t),\gamma(s)) = \frac{|s-t|}{(4R^2+(s-t)^2)^{3/2}}. \tag{11}$$

Then (11) shows that condition (b) of Theorem 6 holds and therefore there is some interval $I=[t_1,t_2]$ such that $\operatorname{supp}\lambda_A=\gamma(I)\cap A_+$. Furthermore, from (10) we see that $W^{\lambda_A}(R+it)$ is strictly increasing on $(-\infty,t_1]$ and is strictly decreasing on $[t_2,\infty)$. By Lemma 2, we can take $I=[c,d]$ and so $\operatorname{supp}\lambda_A=A_+$. □

Even in the case when A is a circle in H^+ (so that $\Gamma(A)$ is a torus in $\mathbb{R}^3$), it is difficult to directly verify the hypothesis (b) of Theorem 6. We next develop sufficient conditions for (b) that, at least in the case A is a circle, are relatively simple to verify.

For $w \in H^+$ and $t \in [a,b]$, let $r_w(t) := |\gamma(t) - w|$, and $s_w(t) := r_w(t) + r_{w_*}(t)$. Assuming γ is twice differentiable at t we have

$$\frac{d^2}{dt^2} K(\gamma(t), w) = \frac{-s''_w(t)s_w(t) + s'_w(t)^2}{s_w(t)^2} \qquad (t \in [a,b]). \tag{12}$$

Then for fixed w, we have that $K(\gamma(t), w)$ is strictly convex on any interval where $s''_w < 0$. Let $u_w(t)$ denote the unit vector $(\gamma(t)-w)/r_w(t)$. Differentiating the dot product $r_w(t)^2 = (\gamma(t)-w)\cdot(\gamma(t)-w)$ we obtain

$$\begin{aligned} r'_w(t) &= \gamma'(t) \cdot u_w(t), \\ u'_w(t) &= (\gamma'(t) - (\gamma'(t) \cdot u_w(t))u_w(t)) / r_w(t), \text{ and} \qquad (13) \\ r''_w(t) &= \gamma''(t) \cdot u_w(t) + \left(|\gamma'(t)|^2 - (\gamma'(t) \cdot u_w(t))^2\right) / r_w(t). \qquad (14) \end{aligned}$$

In the event that γ is parametrized by arclength the above equations can be simplified. In this case $|\gamma'(t)| = 1$. We further assume that $\gamma''(t) \neq 0$ for any $t \in [a,b]$. Then $T(t) = \gamma'(t)$ denotes the unit tangent vector, $\kappa(t) = |T'(t)|$ denotes the curvature, and $N(t) = T'(t)/|T'(t)| = \gamma''(t)/\kappa(t)$ denotes the unit normal vector to the curve γ for $t \in [a,b]$. Substituting these expressions into (13) and (14) we obtain

$$\begin{aligned} r''_w(t) &= \gamma''(t) \cdot u_w(t) + \gamma'(t) \cdot u'_w(t) \qquad (15) \\ &= (N(t) \cdot u_w(t)) \left[\kappa(t) + \frac{N(t) \cdot u_w(t)}{r_w(t)}\right]. \qquad (16) \end{aligned}$$

From this last representation deduce the following.

Lemma 8. *Let $\gamma : [a,b] \to H^+$ be a twice differentiable curve such that $|\gamma'(t)| = 1$ and $\gamma''(t) \neq 0$ for all $t \in [a,b]$. Suppose that for all $s, t \in [a,b]$, $s \neq t$, and $w \in \{\gamma(s), \gamma(s)_*\}$ we have*

$$N(t) \cdot u_w(t) < 0 \text{ and } \left[\kappa(t) + \frac{N(t) \cdot u_w(t)}{r_w(t)}\right] > 0. \tag{17}$$

Then γ satisfies hypothesis (b) of Theorem 6.

We now apply Lemma 8 to the case when A_+ is a subset of a circle.

Corollary 9. *Suppose $C \subset \mathbb{C}$ is a circle of radius $r > 0$ and center a with $\mathrm{Re}[a] > 0$ and suppose A is a compact set in H^+ such that $A_+ \subset C_+$. Then $\operatorname{supp} \lambda_A = A_+^\theta := A_+ \cap \{a + re^{it} \mid |t| \leq \theta\}$ for some $\theta \in [0, \pi/2]$. In particular, if A_+ is a circular arc contained in C_+, then so is $\operatorname{supp} \lambda_A$; consequently, $\operatorname{supp} \mu_{\Gamma(A)}$ is connected.*

Remark: In the case when $\Gamma(A)$ is a torus (that is, if $A = C$), it follows from Corollary 9 that $\operatorname{supp} \mu_{\Gamma(A)}$ is a connected strip of $\Gamma(A)$ of the form $\Gamma(C_+^\theta)$ for some $\theta \in [0, \pi/2]$.

Proof. Without loss of generality we may assume that C has radius $r = 1$ and center $a = R$ for some $R > 0$. We then consider the parametrization of C given by $\gamma(t) := a + e^{it}$ for $t \in [-\pi/2, \pi/2]$. By direct calculation (assisted by Mathematica) we find, for $w = \gamma(s)$,

$$N(t) \cdot u_w(t) = -\left|\sin \frac{s-t}{2}\right| \text{ and } \left[\kappa(t) + \frac{N(t) \cdot u_w(t)}{r_w(t)}\right] = \frac{1}{2},$$

and for $w = \gamma(s)_*$ we find

$$N(t) \cdot u_w(t) = -\frac{2R\cos t + \cos(s+t) + 1}{\sqrt{(2R + \cos s + \cos t)^2 + (\sin s - \sin t)^2}}$$

and

$$\left[\kappa(t) + \frac{N(t) \cdot u_w(t)}{r_w(t)}\right] = \frac{1}{2} + \frac{2R(R + \cos s)}{(2R + \cos s + \cos t)^2 + (\sin s - \sin t)^2}.$$

Then it is easy to verify that the inequalities (17) hold for both $w = \gamma(s)$ and for $w = \gamma(s)_*$ for all $s, t \in [-\pi/2, \pi/2]$ with $s \neq t$. $\square$

5. Kernel in limit $R \to \infty$

One might well conjecture looking at Figure 1 and in light of Theorem 5 or Corollary 7 that for the case of the circle $A = \{z \mid |z-R| = 1\}$, $R > 0$, the support of λ_A is the right-half circle A_+, or equivalently, that the support of the equilibrium measure on the torus $\Gamma(A)$ is the portion of its surface with positive curvature. However, as we see in the limiting case $R \to \infty$, this is not correct.

Define the kernels $K_R : H^+ \times H^+ \to \mathbb{R}$, $R > 0$, and $K_\infty : H^+ \times H^+ \to \mathbb{R}$ by

$$K_R(z, w) := 2R\left(K(R+z, R+w) + \log R\right), \tag{18}$$

$$K_\infty(z, w) := -(\mathrm{Re}[z - w_*] + |z - w|). \tag{19}$$

Using

$$\frac{|z-w| + |2R + z - w_*|}{2R} = 1 + \frac{\mathrm{Re}[z - w_*] + |z - w|}{2R} + \mathcal{O}(R^{-2})$$

we obtain

$$\begin{aligned} K_R(z, w) &= -2R \log \frac{|z-w| + |2R + z - w_*|}{2R} \\ &= -2R \log \left(1 + \frac{\mathrm{Re}[z - w_*] + |z - w|}{2R} + \mathcal{O}(R^{-2})\right) \\ &= -(\mathrm{Re}[z - w_*] + |z - w|) + \mathcal{O}(R^{-1}) \end{aligned}$$

and hence

$$\lim_{R\to\infty} K_R(z,w) = K_\infty(z,w),$$

where the convergence is uniform on compact subsets of $H^+ \times H^+$. We let $J_{K_R}(\mu)$ and $J_{K_\infty}(\mu)$ denote the associated energy integrals defined for compactly supported measures $\mu \in \mathcal{M}(H^+)$.

From the definition of K_R we see that the equilibrium measure λ_A^R on a compact set $A \subset H^+$ with respect to the kernel K_R is equal to $\lambda_{A+R}(\cdot + R)$, that is, $\lambda_A^R(B) = \lambda_{A+R}(B+R)$ where, for a set $B \subset H^+$ and $R > 0$, $B + R$ denotes the translate $\{b + R \mid b \in B\}$.

5.1. **The existence and uniqueness of an equilibrium measure for K_∞.** The weak-star compactness of $\mathcal{M}(A)$ and the continuity of J_{K_∞} imply the existence of a measure $\lambda_A^\infty \in \mathcal{M}(A)$ such that $J_{K_\infty}(\lambda_A^\infty) = \inf_{\mu\in\mathcal{M}(A)} J_{K_\infty}(\mu)$.

We follow arguments developed in [2] to prove the uniqueness of λ_A^∞. First, note that $K_\infty(z,w) = -k_1(z,w) - k_2(z,w)$ where $k_1(z,w) := |z-w|$ and $k_2(z,w) = \mathrm{Re}[z] + \mathrm{Re}[w]$ and so

$$J_{K_\infty}(\mu) = -I_1^*(\mu) - I_2^*(\mu),$$

where I_1^* and I_2^* are the energy integrals associated with the kernels k_1 and k_2, respectively. We need the following lemma of Frostman ([6], also see [2, Lemma 1]).

Lemma 10. *Suppose ν is a compactly supported signed Borel measure on H^+ such that $\int d\nu = 0$ and $I_1^*(\nu) \geq 0$. Then $\nu \equiv 0$.*

For compactly supported Borel measures μ and ν on H^+, let

$$J_{K_\infty}(\mu,\nu) := \iint K_\infty(z,w)\, d\mu(z)\, d\nu(w).$$

Lemma 11. *Suppose A is a compact set in H^+ and $\mu^* \in \mathcal{M}(A)$ satisfies $J_{K_\infty}(\mu^*) = \inf_{\mu\in\mathcal{M}(A)} J_{K_\infty}(\mu)$. For any signed Borel measure ν with support contained in A such that $\nu(A) = \int_A d\nu = 0$ and $\mu^* + \nu \geq 0$, we have $J_{K_\infty}(\mu^*,\nu) \geq 0$.*

Proof. With ν and μ^* as above, we have $\mu^* + \epsilon\,\nu \in \mathcal{M}(A)$ for $0 \leq \epsilon \leq 1$ and so

$$J_{K_\infty}(\mu^*) \leq J_{K_\infty}(\mu^* + \epsilon\,\nu) = J_{K_\infty}(\mu^*) + 2\epsilon J_{K_\infty}(\mu^*,\nu) + \epsilon^2 J_{K_\infty}(\nu). \tag{20}$$

Since (20) holds for all $0 \leq \epsilon \leq 1$, then $J_{K_\infty}(\mu^*,\nu) \geq 0$. □

Theorem 12. *Suppose A is a compact set in the interior of H^+. There is a unique equilibrium measure λ_A^∞ minimizing $J_{K_\infty}(\mu)$ over all $\mu \in \mathcal{M}(A)$. The support of λ_A^∞ is contained in A_+. Furthermore, λ_A^R converges weak-star to λ_A^∞ as $R \to \infty$.*

Remark: Recall that λ_A^R converges *weak-star* to λ_A^∞ (and we write $\lambda_A^R \xrightarrow{*} \lambda_A^\infty$) as $R \to \infty$ means that

$$\lim_{R\to\infty} \int_A f\, d\lambda_A^R = \int_A f\, d\lambda_A^\infty$$

for any function f continuous on A.

Proof. Suppose μ^* and $\tilde{\mu}^*$ are measures in $\mathcal{M}(A)$ such that $J_{K_\infty}(\mu^*) = J_{K_\infty}(\tilde{\mu}^*) = \inf_{\mu\in\mathcal{M}(A)} J_{K_\infty}(\mu)$. Then $\nu := \tilde{\mu}^* - \mu^*$ satisfies the hypotheses of Lemma 11 and thus $J_{K_\infty}(\mu^*, \nu) \geq 0$. On the other hand,

$$J_{K_\infty}(\tilde{\mu}^*) = J_{K_\infty}(\mu^* + \nu) = J_{K_\infty}(\mu^*) + 2J_{K_\infty}(\mu^*, \nu) + J_{K_\infty}(\nu),$$

which, since $J_{K_\infty}(\mu^*) = J_{K_\infty}(\tilde{\mu}^*)$, implies that $J_{K_\infty}(\nu) = -2J_{K_\infty}(\mu^*, \nu) \leq 0$. Now, $J_{K_\infty}(\nu) = -I_1^*(\nu) - I_2^*(\nu) = -I_1^*(\nu)$ since

$$I_2^*(\nu) = \iint (\mathrm{Re}[z] + \mathrm{Re}[w])\, d\nu(z)\, d\nu(w) = 0.$$

Hence, $I_1^*(\nu) = -J_{K_\infty}(\nu) = 2J_{K_\infty}(\mu^*, \nu) \geq 0$ and so, by Lemma 10, it follows that $\nu \equiv 0$ and thus $\mu^* = \tilde{\mu}^*$.

The fact that $\operatorname{supp} \lambda_A^\infty \subset A_+$ follows from the observation that $K_\infty(z, w)$ is strictly decreasing for z varying along all horizontal rays $[iy, \infty + iy)$ for $y \neq \mathrm{Im}[w]$ and along the ray $[w, \infty + iv)$ for $v = \mathrm{Im}[w]$, and is constant along the line segment $[iv, w]$.

The weak-star convergence of λ_A^R to λ_A^∞ follows from the weak-star compactness of $\mathcal{M}(A)$ and the uniqueness of the equilibrium measure λ_A^∞. □

Remarks:

(1) The level sets of $K_\infty(\cdot, w)$ are parabolas with focus w and directrix $x = a$ for $a > \mathrm{Re}[w]$ (in the case $a = \mathrm{Re}[w]$, the level set is the line segment $[iv, w]$ where $v = \mathrm{Im}[w]$). Notice that these parabolas can also be viewed as arising from the elliptical level curves illustrated in Figure 3 by letting the real part of the focus w_* tend to $-\infty$.

(2) One may also consider $K_\infty(z, w)$ on $\mathbb{C}\times\mathbb{C}$ rather than $H^+ \times H^+$ (in effect, the line $\mathrm{Re}[z] = -\infty$ may be considered the axis of rotation).

Let W_∞^μ denote the potential for a measure $\mu \in \mathcal{M}(H^+)$ and kernel K_∞:

$$W_\infty^\mu(z) = \int_A K_\infty(z, w)\, d\mu(w) \qquad (z \in H^+).$$

Then W_∞^μ is continuous on H^+. Furthermore, if $W_\infty^\mu(z)$ is not constant for $z \in \operatorname{supp}\mu$, then one may construct a signed Borel measure ν with

support contained in A such that $\nu(A) = \int_A d\nu = 0$, $\mu+\nu \geq 0$, and such that $J_{K_\infty}(\mu,\nu) < 0$ (cf. [2]). Lemma 11 then implies that $J_{K_\infty}(\mu,\nu)$ cannot be minimal, which gives the following result.

Lemma 13. *The equilibrium potential $W_\infty^{\lambda_A^\infty}$ satisfies*

$$W_\infty^{\lambda_A^\infty}(z) \geq J_{K_\infty}(\lambda_A^\infty) \qquad (z \in A) \tag{21}$$

with equality if $z \in \operatorname{supp} \lambda_A^\infty$.

5.2. **Properties of the equilibrium measure for a circle.** We next consider the support of the K_∞-equilibrium measure in the case that A_+ is contained in the right-half of a circular arc (as in Corollary 9). Recall that if C is the circle with center a and radius r and $B \subset C$, we define $B^\theta := B \cap \{a + re^{it} \mid -\theta \leq t \leq \theta\}$.

Theorem 14. *Suppose $C \subset \mathbb{C}$ is a circle of radius $r > 0$ and center a with $\operatorname{Re}[a] > 0$ and suppose A is a non-empty compact set in H^+ such that $A_+ \subset C_+$. Then $\operatorname{supp} \lambda_A^\infty = A_+^\theta$ for some $\theta \in [0, \pi/2]$.*

Furthermore, if A_+ is also symmetric about the line $y = \operatorname{Im}[a]$ and $A_+^{\pi/3}$ is non-empty, then $\operatorname{supp} \lambda_A^\infty = A_+^\theta$ for some $\theta \in [0, \pi/3]$. Moreover, if A_+ is also symmetric about the line $y = \operatorname{Im}[a]$ and $A_+^{\pi/3}$ is empty, then $\lambda_A^\infty = (\delta_{a+\zeta} + \delta_{a+\overline{\zeta}})/2$ where $\zeta := re^{i\theta_m}$ and $\theta_m := \min\{\theta \geq 0 \mid a + re^{i\theta} \in A_+\}$.

Proof. Without loss of generality we may assume that C has radius $r = 1$ and center $a = 0$. We then consider the parametrization of C given by $\gamma(t) := e^{it}$ for $-\pi/2 \leq t \leq \pi/2$. Then, using $|e^{it} - e^{is}| = 2\,|\sin((s-t)/2)|$, we find

$$K_\infty(\gamma(t),\gamma(s)) = -\cos(t) - \cos(s) - 2\left|\sin\frac{s-t}{2}\right| \qquad (s,t \in [-\pi/2, \pi/2]).$$

Differentiating twice with respect to s we obtain

$$\frac{\partial^2}{\partial t^2} K_\infty(\gamma(t),\gamma(s)) = \frac{1}{2}\left|\sin\frac{s-t}{2}\right| + \cos(t)$$

which is positive for $-\pi/2 < s, t < \pi/2$. Then (as in the proof of Corollary 9) it follows that $\operatorname{supp} \lambda_A = A_+^\theta$ for some $\theta \in [0, \pi/2]$.

Now suppose A_+ is symmetric about the x-axis. Then the uniqueness of λ_A^∞ shows that λ_A^∞ is also symmetric about the x-axis, that is, $d\lambda_A^\infty(w) = d\lambda_A^\infty(\overline{w})$ for $w \in H^+$. Thus we have

$$W_\infty^{\lambda_A^\infty}(\gamma(t)) \;=\; \int_{A_+} K_\infty^s(z,w)\, d\lambda_A^\infty(w),$$

where

$$K_\infty^s(z,w) := \left(K_\infty(z,w) + K_\infty(z,\overline{w})\right)/2 \qquad (z,w \in H^+).$$

Then we have

$$K_\infty^s(\gamma(t),\gamma(s)) \\ = \begin{cases} -\cos(s) - \cos(t) - 2\cos(s/2)\sin(t/2)\,, & 0 \le s < t \le \pi/2, \\ -\cos(s) - \cos(t) - 2\cos(t/2)\sin(s/2)\,, & 0 \le t < s \le \pi/2. \end{cases}$$

and differentiating with respect to t we obtain

$$(22) \qquad \frac{\partial}{\partial t} K_\infty^s(\gamma(t),\gamma(s)) \\ = \begin{cases} \sin(t) - \cos(s/2)\cos(t/2)\,, & 0 \le s < t \le \pi/2, \\ \sin(t) + \sin(s/2)\sin(t/2)\,, & 0 \le t < s \le \pi/2. \end{cases}$$

We claim that

$$(23) \qquad \frac{\partial}{\partial t} K_\infty^s(\gamma(t),\gamma(s)) > 0 \qquad (-\pi/2 \le s \le \pi/2,\ t > \pi/3).$$

Clearly (23) holds in the second case of (22) when $0 < t < s \le \pi/2$. If $\pi/3 < t \le \pi/2$ and $0 \le s < t$, then using the first case of (22),

$$\sin(t) - \cos(s/2)\cos(t/2) = \cos(t/2)\left(2\sin(t/2) - \cos(s/2)\right)$$

and $2\sin(t/2) - \cos(s/2) \ge 2\sin(t/2) - 1 > 0$ for this range of s and t, we see that (23) holds in this case as well. Hence, we have

$$\frac{d}{dt} W_\infty^{\lambda_A^\infty}(\gamma(t)) = \int_{A_+} \frac{\partial}{\partial t} K_\infty^s(\gamma(t),w)\, d\lambda_A^\infty(w) > 0 \qquad (t > \pi/3).$$

Thus, in light of Lemma 13, we have $\operatorname{supp} \lambda_A^\infty \subset A_+^{\pi/3}$ if $A_+^{\pi/3} \ne \emptyset$, while if $A_+^{\pi/3} = \emptyset$, then $\lambda_A^\infty = (\delta_{a+\zeta} + \delta_{a+\overline{\zeta}})/2$. □

5.3. **The vertical line segment.** In this section we consider sets $A \subset H^+$ such that A_+ is contained in a vertical line segment $[a+ic, a+id]$ and further suppose the endpoints $a+ic$ and $a+id$ are in A_+. Then

$$K_\infty(a+it, a+is) = -2a - |t-s| \qquad (s,t \in [c,d])$$

which falls into the class of kernels studied in [2] and it follows from results there that $\lambda_A^\infty = \left(\delta_{a+ic} + \delta_{a+id}\right)/2$ where δ_w denotes the unit point mass at w. In particular, for the "infinite washer" in $\mathbb{R}^3$ obtained by rotating $[a+ic, a+id]$ about the y-axis and letting $a \to \infty$, the support of the equilibrium measure degenerates to two circles. We contrast this with the finite R case where, by Corollary 7, we have $\operatorname{supp} \lambda_A^R = A_+$.

6. Discrete Minimum Energy Problems on $A \subset H^+$

Suppose $A \subset H^+$ is compact, $k : A \times A \to \mathbb{R}_+$ is continuous and nonnegative, and that there is a unique equilibrium measure $\lambda_{k,A}$ minimizing the k-energy

$$I_k(\mu) := \iint_{A\times A} k(x,y)\; d\mu(x)\, d\mu(y)$$

over measures $\mu \in \mathcal{M}(A)$. In this case we say that k is a *continuous admissible kernel on* A. In particular, we have in mind the reduced kernel K as defined in (5) or the limiting kernel K_∞ as defined in (19).

We consider the following discrete minimum k-energy problem. The arguments in this section closely follow those in [11, pp. 160–162]; however, the continuity of k here allows for some simplification. For a collection of $N \geq 2$ distinct points $\omega_N := \{x_1, \ldots, x_N\} \subset A$, let

$$E_k(\omega_N) := \sum_{1\leq i\neq j\leq N} k(x_i, x_j) = \sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} k(x_i, x_j),$$

and

$$\mathcal{E}_k(A,N) := \inf\{E_k(\omega_N) \mid \omega_N \subset A, |\omega_N| = N\}. \tag{24}$$

Since

$$\mathcal{E}_k(A,N) \leq \sum_{1\leq i\neq j\leq N} k(x_i, x_j) \tag{25}$$

for any configuration of N points $\{x_1, \ldots, x_N\} \subset A$, integrating (25) with respect to $d\lambda_{k,A}(x_1) d\lambda_{k,A}(x_2) \cdots d\lambda_{k,A}(x_N)$ we find $\mathcal{E}_k(A,N) \leq N(N-1) I_k(\lambda_{k,A})$ and so we have

$$\frac{\mathcal{E}_k(A,N)}{N(N-1)} \leq I_k(\lambda_{k,A}) \qquad (N \geq 2). \tag{26}$$

On the other hand, the compactness of A and continuity of k imply that for each $N \geq 2$ there exists some *optimal k-energy configuration* $\omega_N^* \subset A$ such that $E_k(\omega_N^*) = \mathcal{E}_k(A,N)$. Let $\lambda_{A,N} = \frac{1}{N}\sum_{x\in\omega_N^*} \delta_x \in \mathcal{M}(A)$ (where δ_x denotes the unit point mass at x). Then

$$I_k(\lambda_{k,A}) \leq I_k(\lambda_{A,N}) = \frac{\mathcal{E}_k(A,N) + \sum_{i=1}^{N} k(x_i, x_i)}{N^2} \qquad (N \geq 2). \tag{27}$$

Combining (26) and (27) we have

$$\frac{\mathcal{E}_k(A,N)}{N(N-1)} \leq I_k(\lambda_{k,A}) \leq I_k(\lambda_{A,N}) \leq \frac{\mathcal{E}_k(A,N)}{N^2} + \frac{\|k\|_A}{N} \quad (N \geq 2), \tag{28}$$

where $\|k\|_A := \sup_{z\in A} k(z,z)$. Since $\mathcal{E}_k(A,N)/N^2 \le I_k(\lambda_{k,A}) < \infty$, the inequalities in (28) show that there is some constant C such that $0 \le I_k(\lambda_{A,N}) - I_k(\lambda_{k,A}) \le C/N$ for $N \ge 2$, and so

$$I_k(\lambda_{A,N}) \to I_k(\lambda_{k,A}) \text{ as } N \to \infty. \tag{29}$$

If μ^* is a weak-star limit point of the sequence $\{\lambda_{A,N}\}$, then (29) shows that $I_k(\mu^*) = I_k(\lambda_{k,A})$ and so $\mu^* = \lambda_{k,A}$. By the weak-star compactness of $\mathcal{M}(A)$, any subsequence of $\{\lambda_{A,N}\}$ must contain a weak-star convergent subsequence. Hence, we have the following result.

Proposition 15. *Suppose A is a compact set in H^+ and that $k : A\times A \to \mathbb{R}_+$ is a continuous admissible kernel on A. For $N \ge 2$, let ω_N^* be an optimal k-energy configuration of N points $\{x_1, x_2, \ldots, x_N\} \subset A$. Then $\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \xrightarrow{*} \lambda_{k,A}$ as $N \to \infty$.*

Figure 4 shows (near) optimal K-energy configurations for $N = 30$ points restricted to various ellipses in H^+.

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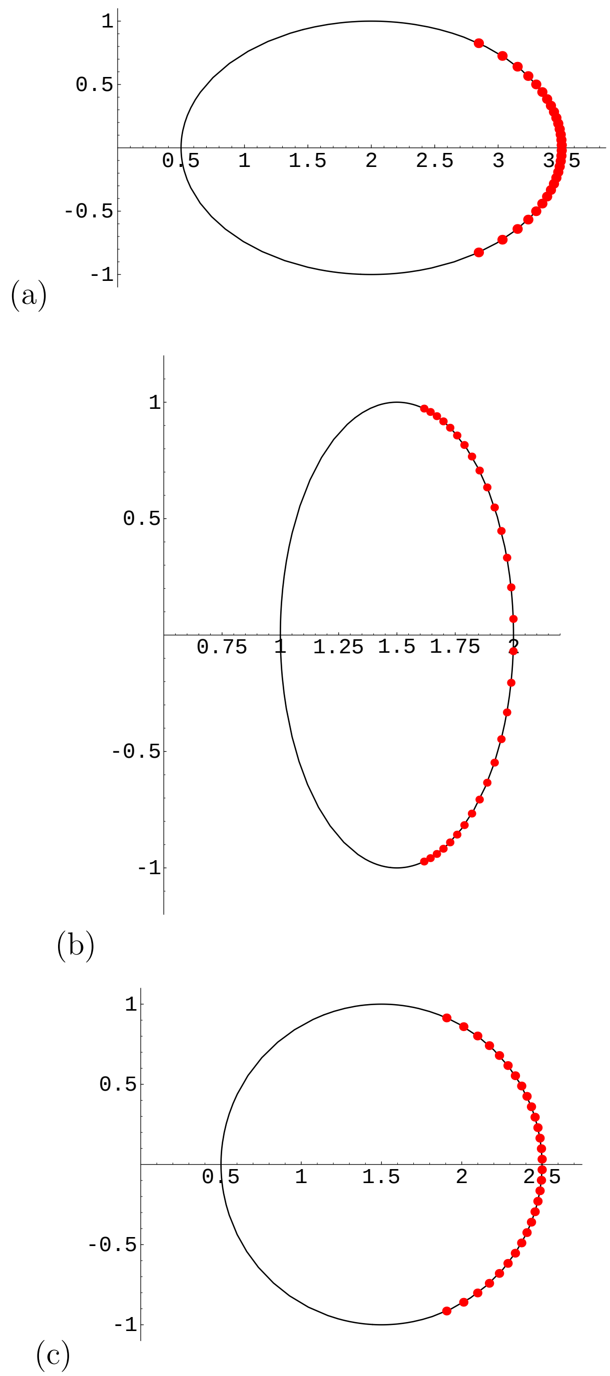

FIGURE 4. Near optimal K-energy configurations ($N = 30$ points) on various ellipses in H^+.

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